

Elucidating Discount Function Properties from a Filtering Perspective

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ABSTRACT

This study identifies two properties of discount functions. First, discount functions can be conceived as nonlinear filters, where dynamic discount functions are low pass filters and static discount functions are high pass filters. Second, whether a discount function is static or dynamic depends on how interest rate paths are determined. Since the general principles of discounting and pricing apply to many financial products, the findings of this study are significant to financial theory and practice.

Keywords: *The Yule-Slutsky effect, low pass filter, stochastic pricing models, annuity*

INTRODUCTION

Interest rate is a major determinant of theoretical prices of financial products. Some financial institutions have been developing new products by combining valuation models with innovative techniques where the specification of discount functions and interest rate dynamics provide the basis of model building. This study investigates the properties of discount functions using empirical data and provides fresh perspectives for reconsidering the robustness of economic and financial theories concerning financial products. Because of data availability, this study takes the pricing model of an immediate certain annuity as an example to demonstrate how assumptions regarding interest rates significantly influence price dynamics. An immediate certain annuity is a periodic payment that commences after the premium is paid. Verifying the findings of this study is straightforward, because the calculations are simple and the data sources are accessible on the internet. Since the work of science is to substitute facts for appearances and demonstrations for impressions, theory evaluation should be based on empirical evidence and scientific facts.

Throughout this paper, $PV(t)$ denotes the time series of the present value of an immediate certain annuity comprising \$1 monthly benefit payments for specified years. The first half of this paper comprises a comparative static analysis of $PV(t)$ characteristics using historical U.S. data. The findings do not support the assumption made by stochastic pricing models that $PV(t)$ movement is a stochastic process driven by stochastic interest rates. The second half of this study proposes the existence of two types of discount functions. A dynamic discount function is a low pass filter that removes high frequency noise from input data and yields smooth output data. Meanwhile, a static discount function is a high pass filter that passes the high frequency noise from inputs to outputs. Methodological confusion may lead to mispricing of financial products for which quotes are constructed mathematically.

The remainder of this paper is organized as follows. Section two describes the methods and data used to derive $PV(t)$. Section three then analyzes the empirical findings. Subsequently, section four examines the properties of dynamic discount functions and section five analyzes static discount function properties. Finally, section six draws conclusions.

METHOD AND DATA

Assume a certain annuity pays 1 dollar at the end of each month for N years. The discount function can be expressed as follows:

$$PV(t) = \frac{1}{1+i_1} + \frac{1}{(1+i_1)(1+i_2)} + \frac{1}{(1+i_1)(1+i_2)(1+i_3)} + \dots + \frac{1}{\prod_{k=1}^{12N} (1+i_k)} \quad (1)$$

where i_k denotes the discount rate during the k -th month. Comparative analysis of the effects of assumed and actual interest rates can be performed by comparing $PV(t)$ derived from different interest rates.

Interest rate data was mostly obtained from the website of the Federal Reserve Bank of St. Louis. Two additional sources were the book *A History of Interest Rates* by Homer and Sylla (2005) and the Center for Research in Security Prices. Because Equation (1) is a decreasing function of interest rates, a series of i_k at a low level produces a series of $PV(t)$ at a high level, and vice versa, which is a scaling relationship that does not change the inference validity if different types or levels of interest rates are selected as i_k . This study presents the $PV(t)$ calculated using the yields on Moody's Aaa corporate bonds. Figure 1 illustrates a profile of the annual yields on Moody's Aaa corporate bonds, Moody's Baa corporate bonds, and long term government bonds from 1919 to 2004. The long term trend changed twice during the sample period. As this study focuses on the long-term price behavior, an econometric analysis is inappropriate due to the data undergoing multiple structural changes.

Figure 1 Bond Yields, 1919-2004

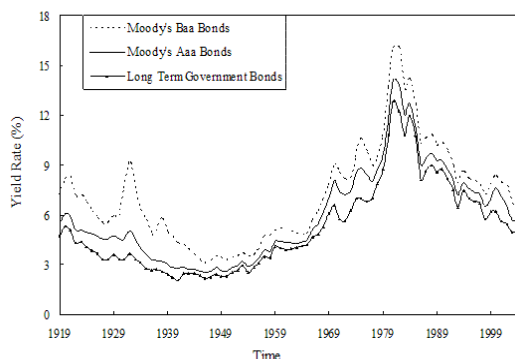
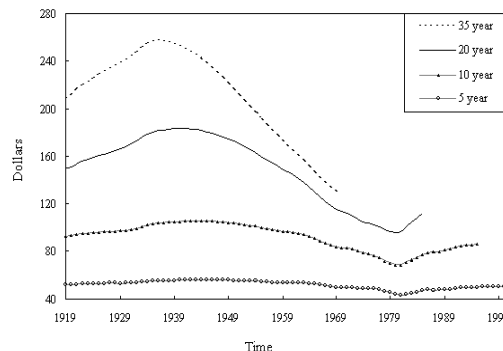


Figure 2 PV(t) of the 5, 10, and 35-Year Certain Annuities



EMPIRICAL FINDINGS

The analysis of the results of the computations involved in Equation (1) comprises three subjects, including the near cycle of $PV(t)$, the effect of interest rate assumptions on $PV(t)$, and the classification of discount functions.

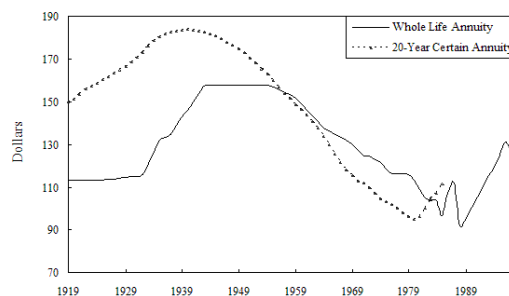
Near Cycle of $PV(t)$

Under constraints of data availability, the number of $PV(t)$ for long annuities is less than that for short annuities. Given a sufficiently large number of $PV(t)$, it is possible to see whether structural changes and cycles exist in $PV(t)$. Figure 2 illustrates the $PV(t)$ for the 5, 10, 20, and 35-year annuities obtained by adjusting the number of summation terms in Equation (1). Figure 2 reveals two characteristics of $PV(t)$

curves. First, both short and long annuities are affected by the interest rate cycle profiled in Figure 1, and that structural changes in $PV(t)$ of all annuities occur in the same year. Second, a long duration of benefit payments increases the number of summation terms in Equation (1), which raises the $PV(t)$ curve and makes the near cycle distinct. Actual premium data substantiates the conjecture regarding near cycles of $PV(t)$. From Figure 3, the solid line approximates the premiums for an immediate whole life annuity for males aged 65, while the dotted line denotes the $PV(t)$ of a 20-year certain annuity. The two lines exhibit similar trends, suggesting that $PV(t)$ may be cyclical over the long term. The explanation for this long cycle is unclear. Two research provide plausible answers to this question, the long wave theory and the Yule-Slutsky effect; the main propositions of which are summarized below.

- (1) The long wave theory examines various influences on economic cycles over approximate durations of 40 to 60 years (Kondratieff cycles), among which the clustering of innovation and technological changes is the most influential driver of long waves during economic development. However, because of the deficiencies of quantitative methods for analyzing long term data, both the existence and the precise timing of long waves are controversial and difficult to evaluate. (Freeman and Louçã, 2001; Louçã and Reijnders, 1999; Maddison, 2007)
- (2) The Yule-Slutsky effect was independently discovered during the 1920s. (Bullock et al, 1927; Slutsky, 1937; Yule, 1926) Two propositions in relation to the effect are as follows. First, the summation of random causes may be the source of undulatory processes. Second, these wave-like movements will exhibit an approximate regularity in sinusoidal form. Economists interpret this effect in two ways, indicating the difficulty in understanding the nature of cyclic processes. First, an unclear mechanism of aggregation on random series causes genuine cycles. Second, data manipulation can generate spurious cycles. (Barnett, 2006; Klein, 1999; Kuznets, 1929; Kydland and Prescott, 1990) In Figures 2, the cyclic characteristic of the $PV(t)$ curves for long annuities are more visible than that for short annuities, supporting the first proposition of the Yule-Slutsky effect. However, the second proposition cannot be verified due to deficiencies in the data.

Figure 3 Premiums and $PV(t)$



Approximate premiums of the whole life annuity for males at age 65 are derived from two sources: Warshawsky (1988) and Best's Policy Reports (1997 and 1998). The premium for \$1 monthly income is the mean premium charged by 10 large U.S. life insurers.

The issue of long cycles is relevant to financial markets because interest rate is procyclical (positively correlated with GDP growth) and PV is countercyclical (negatively correlated with GDP growth). Interest rate dynamics is driven by the real sector of the economy. What provides direct evidence regarding long cycles in the real sector provides circumstantial evidence regarding long cycles in the financial sector. The controversy surrounding long cycles seems to result from the lack of an appropriate model rather than a lack of data, since currently econometric models are incapable of dealing with data

containing multiple structural changes without partitioning the data, and statistical analysis of partitioned data does not make sense when the subject is the long term behaviors of the variables.

Effect of Interest Rate Assumptions

Sometimes the implicit forward rates in the term structure of interest rates are taken as the expected future spot rates. The following is a simple method of computing $PV(t)$ based on forward rates. First, monthly forward rates are derived by linear interpolation and extrapolation for the yields on the 1, 2, 5, 7, 10, 20, and 30-year Treasury securities. Next, monthly forward rates are substituted for i_k in Equation (1). $PV(1942)$ uses the forward rates with the 1942 yield curve, $PV(1970)$ uses the forward rates with the 1970 yield curve, and so on.

Figure 4 displays the term structure of yield rates on U.S. Treasury securities from 1942 to 2004. Figure 5 shows two types of $PV(t)$ of 35-year annuities; the solid curve is derived from forward rates, while the dotted curve is derived from actual spot rates. The two curves may differ due to three factors: (1) Yields on Treasury securities are less than on corporate bonds. (2) Forward rates behave differently to future spot rates. (3) When using yield curves to compute $PV(t)$, it is necessary to replace 12N discount rates as a whole unit following change in the yield curve. In contrast, using actual spot rates means only a few spot rates are regularly replaced and most are retained.

Figure 4 Term Structure of Yields on U.S. Treasury Securities, 1942-2004

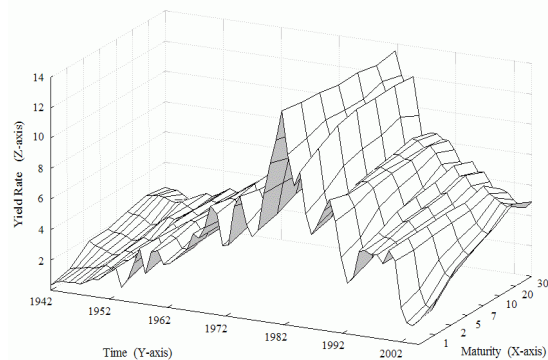
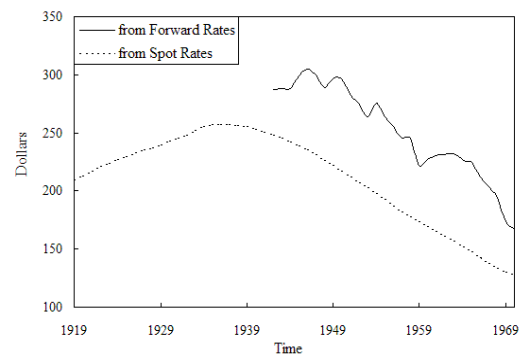


Figure 5 $PV(t)$ Derived from Two Types of Interest Rates



Concerning the second of the above points, three hypotheses regarding the term structure of interest rates, namely, the expectations hypothesis, liquidity preference hypothesis, and market segmentation hypothesis, provide different answers regarding the relationship between forward rates and future spot rates. Besides these answers, another answer can be found by examining the directions of the curves in Figure 4. The spot rates move on the Z-Y plane with fluctuations and structural change, while the yield rates move on the Z-X plane without fluctuations or structural change. Annual shifts of the yield curves on the Z-X plane are different to spot rate movements along the path on the Z-Y plane. In Figure 5 the solid curve is derived from 29 yield curves covering the period from 1942 to 1970, and one shift of the curve leads to the replacement of all 12N discount rates in Equation (1). However, the dotted curve in Figure 5 is derived from only one spot rate path, which requires the regular annual replacement of 12 discount rates by new spot rates while retaining most of them. $PV(t)$ fluctuates as a result of fluctuations of sampling from curve to curve on the Z-X plane rather than fluctuations of spot rates on the same path on the Z-Y plane.

Classification of Discount Functions

In the previous sections the $PV(t)$ curves derived from spot interest rates are smooth and near-cyclical, and fluctuations of interest rates do not cause fluctuations of $PV(t)$. The trend rather than the volatility of interest rates is what affects the dynamics of $PV(t)$. In this respect, a discount function is a low pass filter that allows low frequencies (trend) to pass and removes high frequencies (noise) from inputs.¹ On the other hand, the $PV(t)$ curves derived from forward interest rates exhibit wavelets. In this respect, a discount function resembles a high pass filter that passes high frequencies. Whether the output of a discount function is stochastic or non-stochastic depends on input selection. The relationship between interest rates and $PV(t)$ is nonlinear, and a discount function may be conceived as a nonlinear filter. Since currently no general theory exists capable of handling all nonlinear functions, knowledge of nonlinear functions must be constructed from the input/output patterns of individual cases. (Astola and Kuosmanen, 1997; Bendat, 1998; Mallows, 1980; Pearson, 1999)

The following sections employ the methodology of filtering and classify discount functions into two types, dynamic and static, based on two reasons. First, the filtering technique has been used in economics since the 1980s to decompose macroeconomic time series into trends and noise in research on business cycles. (Baxter and King, 1999; Christiano and Fitzgerald, 2003; Hodrick and Prescott, 1997) Second, in “Mathematics Subject Classification 2010”, the category of stochastic systems includes estimation and detection, filtering, system identification, data smoothing, and so on. Despite conflicting methodologically with stochastic pricing models that suppose $PV(t)$ is a stochastic process driven by stochastic interest rates, filtering is a field within stochastic system theory and should be considered in studying the price behaviors of financial products.

PROPERTIES OF THE DYNAMIC DISCOUNT FUNCTION

This section proposes theses regarding dynamic discount function properties. First, a dynamic discount function is a special Volterra filter. Second, a dynamic discount function is a nonlinear moving average function of interest rates. Third, a dynamic discount function is a recursive filter.

Dynamic Discount Function as a Special Volterra Filter

The Volterra filter comes from simplifying the Volterra functional which is a generalization of the Taylor series expansion of a function. A functional is an operation on a function that yields a number. The Volterra functional is stated as: (Schetzen, 1989)

$$y(t) = \int_{-\infty}^{\infty} h_1(s_1) x(t-s_1) ds_1 + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(s_1, s_2) x(t-s_1)x(t-s_2) ds_1 ds_2 + \dots \quad (2)$$
$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(s_1, s_2, \dots, s_n) x(t-s_1)x(t-s_2) \dots x(t-s_n) ds_1 \dots ds_n + \dots$$

¹ Filters are broadly classified as low pass, high pass, and band pass filters. A high pass filter is a device that passes high frequencies and blocks low frequencies of the inputs. A band pass filter is a device that passes frequencies within a certain band and blocks frequencies outside that band. Filters are used to remove interference, to transform signals into some other form, and to extract some of the signal's characteristics. Further expositions can be found in the literature of the filtering theory and applications.

The discrete form is:

$$y(t) = \sum_{s_1=-k}^k h_1(s_1) x(t-s_1) + \sum_{s_1=-k}^k \sum_{s_2=-k}^k h_2(s_1, s_2) x(t-s_1)x(t-s_2) + \dots$$

$$+ \sum_{s_1=-k}^k \sum_{s_2=-k}^k \dots \sum_{s_n=-k}^k h_n(s_1, s_2, \dots, s_n) x(t-s_1)x(t-s_2) \dots x(t-s_n)$$
(3)

where $x(\cdot)$ denotes the input, $y(\cdot)$ denotes the output, n is the polynomial or nonlinearity order, k is the dynamic order, and $h_n(\cdot)$ represents the impulse response function. Due to the difficulty of estimating $h_n(\cdot)$, in practice a low order version of Equation (3) is specified and termed the Volterra filter or Volterra model. (Astola and Kuosmanen, 1997; Mathews and Sicuranza 2000; Pearson, 1999) The IEEE electronic library contains literature on the Volterra filters. The dynamic discount function of an annuity is a special Volterra filter due to the following:

- (1) $\sum_{s_1=-k}^k \sum_{s_2=-k}^k \dots \sum_{s_n=-k}^k h_n(s_1, s_2, \dots, s_n) x(t-s_1)x(t-s_2) \dots x(t-s_n)$ is the n th-order Volterra operator. For each Volterra operator, if most of the $h_n(\cdot)$ are 0, but $h_n(\cdot) \neq 0$ at $s_1 = 0, s_2 = -1, \dots, s_n = -n + 1$, that is,

$$\sum_{s_1=-k}^k \sum_{s_2=-k}^k \dots \sum_{s_n=-k}^k h_n(s_1, s_2, \dots, s_n) x(t-s_1)x(t-s_2) \dots x(t-s_n)$$

$$= h_n(0, -1, \dots, -n + 1) x(t) x(t+1) \dots x(t+n-1)$$
(4)

thus Equation (3) can be reduced to a sparse or pruned Volterra model:

$$y(t) = h_1(\cdot) x(t) + h_2(\cdot) x(t) x(t+1) + \dots + h_n(\cdot) x(t) x(t+1) \dots x(t+n-1)$$
(5)

- (2) Let $h_n(\cdot)$ denote the benefit payment, $x(t+n-1)$ the discount factor of the n -th period, and $h_n(\cdot) x(t) x(t+1) \dots x(t+n-1)$ the discounted value of the benefit payment of the n -th period.

Equation (5) is then a discount function of a certain annuity.

- (3) Let $h_n(\cdot)$ be the survival rate of the n -th period, and $x(t+n-1)$ represent the discount factor. Equation (5) is then a discount function of a life annuity with fixed benefit payments.

Dynamic Discount Function as a Nonlinear Moving Average Function of Interest Rates

A moving average function of x_t can be expressed in general form as:

$$y_t = f(x_{t-k}, \dots, x_{t-1}, x_t, x_{t+1}, \dots, x_{t+k})$$
(6)

In the terminology of filtering, $f(\cdot)$ is a moving window filter with a window length of $2k+1$. The larger the window length, the smoother the output y_t . A moving average function is one of the low pass filters that can remove or attenuate fluctuations of the inputs and yield smooth outputs. The degree of smoothing depends on the specifications of $f(\cdot)$. In most time series models, $f(\cdot)$ is linear and x_t are data of the past and error terms, while for a dynamic discount function $f(\cdot)$ is nonlinear and x_t represents expected future interest rates or discount factors.

Let y_t denote the present value at time t , i_k the current and future interest rates, and $x_k = \frac{1}{1+i_k}$

the discount factors, $t \leq k \leq t+n-1$. The discount function of a certain annuity with n fixed payments can be stated as:

$$y_t = x_t + x_t x_{t+1} + \dots + x_t x_{t+1} \dots x_{t+n-1} = \frac{1}{1+i_t} + \frac{1}{1+i_t} \cdot \frac{1}{1+i_{t+1}} + \dots + \prod_{k=t}^{t+n-1} \frac{1}{1+i_k}$$
(7)

where y_t is the nonlinear moving average of i_k . For y_t and y_{t-1} , the number of inputs in common is $n-1$. That number increases with decreasing time interval of discounting, which enhances the correlation between y_t and y_{t-1} . Although i_k is independently and identically distributed, $\Delta y_t = y_t - y_{t-1}$ is probably not an *iid* variable. By definition, y_t does not exhibit Brownian motion if Δy_t is not an *iid* variable.

Some indices in the chart analysis of stock prices provide examples of the Yule-Slutsky effect and the smoothing result from moving averaging. KD, RSI, and MACD are indices derived by moving average or moving summation functions. The curves of KD, RSI, and MACD display two features. First, functions with different specifications can transform stochastic prices into spurious cycles with consistent timing. Second, window length of the function rather than volatility of input data is the major determinant of the smoothness of the curves. The existence of the Yule-Slutsky effect challenges the methodology of stochastic pricing models.

Dynamic Discount Function as a Recursive Filter

In section 3.1 Near Cycle of PV(t), Figure 2 shows that the increased duration of a certain annuity lifts the PV(t) curve without changing its trend. A certain annuity becomes a perpetual annuity if its duration increases infinitely. The discount function of the perpetual annuity provides an example of the recursive filter.

Assume a perpetual annuity pays 1 dollar at the end of each period. Let i_t denote the discount rate and $x_t = \frac{1}{1+i_t}$ represent the discount factor in the t -th period. Then the time series of the present value y_t can be stated as:

$$y_t = x_t + x_t x_{t+1} + x_t x_{t+1} x_{t+2} + x_t x_{t+1} x_{t+2} x_{t+3} + \dots \quad (8)$$

$$y_{t+1} = x_{t+1} + x_{t+1} x_{t+2} + x_{t+1} x_{t+2} x_{t+3} + x_{t+1} x_{t+2} x_{t+3} x_{t+4} + \dots$$

$$y_{t+1} = \frac{y_t}{x_t} - 1 = y_t(1+i_t) - 1 = y_t + y_t i_t - 1 \quad (9)$$

$$\Delta y_{t+1} = y_{t+1} - y_t = y_t i_t - 1 \quad (10)$$

For Equation (10), $y_t = \frac{1}{i_t}$ makes $\Delta y_{t+1} = 0$, which is a special condition resulting in a static

discount function. Equation (9) has the following properties:

- (1) It can be calculated recursively and yields highly correlated outputs since y_t is used directly to calculate y_{t+1} . A recursive filter has better immunity to impulses than a nonrecursive one, and its outputs are characterized by either exponentially growing, decaying, or sinusoidal signal output components. (Astola and Kuosmanen, 1997; Smith, 2006)
- (2) Equation (9) is a bilinear function; that is, it is a function of two variables, and is linear in the parameters and nonlinear in the product terms involving i_t and y_t . Bilinear functions comprise the simplest class of nonlinear functions.
- (3) A Volterra series representing a nonlinear system with memory or expectations, such as Equation (8), has a bilinear realization which can simplify the computations involved in estimation and control. (Bruni et al, 1974; Elliott, 1999; Mohler and Kolodziej, 1980; Mathews and Sicuranza, 2000)

The next section uses actual interest rate data to examine the input/output relationship of Equation (9) and verify its filtering property.

PROPERTIES OF THE STATIC DISCOUNT FUNCTION

This section proposes that using static interest rate models causes fluctuations of PV(t). Static models conceive economic activity as a sequence of snapshots, each representing a market equilibrium or rest point. A yield curve is a static interest rate path representing the equilibrium of a bond market during one or a few days, and the forward rates derived from a yield curve represent a current forecast of future spot rates. Devoid of history, a nominally dynamic stochastic process involves complex repetition rather than dynamic development and thus is inherently static. This section comprises two parts. The first part compares the filtering property between the dynamic and static discount functions. The second part examines the static nature of stochastic models.

Comparing Input/Output Patterns between Static and Dynamic Discount Functions

To simplify the calculation, the present value of a stream of constant cash flow is determined by assuming a fixed interest rate i , and Equation (8) is reduced to

$$y = \frac{1}{i} \quad \text{or} \quad y_t = \frac{1}{i_t} \tag{11}$$

Equation (11) is nested within Equation (9), as $y_t i_t = 1$ in Equation (11) validates $\Delta y_{t+1} = 0$ in Equation (10). In Equation (11), i denotes the level of a flat interest rate path; a change of interest rate is equivalent to a horizontal upwards or downwards shift in interest rate path, which assigns a new value to y . Owing to its simplicity, this static approach has a long history of practical use in pricing government bonds and preferred stocks that are perpetual and paying fixed interest rates. (Homer and Sylla, 2005)

The outputs of Equations (9) and (11) are compared using two sets of interest data, the actual spot rate, denoted as i_t and its trend, denoted as i_t^* . i_t is the monthly yield on Moody's Aaa corporate bonds from 1919 to 2004, $\frac{2.46\%}{12} \leq i_t \leq \frac{15.94\%}{12}$, $t = 1, 2, \dots, 1032$. Removing the short term volatilities of i_t using the Hodrick-Prescott Filter built into the software EViews yields i_t^* . The output becomes smoother with increasing smoothing parameter of the Hodrick-Prescott Filter. (Hodrick and Prescott, 1997) To obtain a very smooth curve of i_t^* , the value of the smoothing parameter for monthly data is increased from the default of 14400 to 144000. Figure 6 shows the graphs of $12i_t$ and $12i_t^*$. With i_t and i_t^* being the inputs in Equation (11), the outputs are y_t and y_t^* , respectively, and are graphed in Figure 7. Comparing Figures 6 and 7 reveals two main features. First, the behavior of present values is an approximate mirror image of that of interest rates. Second, the present value is sensitive to interest rate volatility, as smooth i_t^* produces smooth y_t^* , and fluctuating i_t produces fluctuating y_t .

To study the input/output pattern of Equation (9), it is necessary to assume appropriate initial values² for y_t , two of which are 264.3 and 264.6. Figure 8 graphs y_t , the input of which is i_t . Meanwhile, Figure 9 graphs y_t^* , the input of which is i_t^* . y_t and y_t^* do not fluctuate in the short term but swing smoothly over the long term. Additionally, the trajectories of y_t and y_t^* are sensitive to the initial values, which is not unusual in the case of nonlinear functions. An inference drawn from Figures 6 to 9 is as follows. For a static function such as Equation (11), y_t is sensitive to the volatility of i_t ; for a

² For 1919, the present values for the 60, 70, 80, and 90-year certain annuities with monthly payments of 1 dollar are 256.48, 260.86, 262.48, and 263.28, respectively. These numbers are referenced when selecting appropriate initial values for y_t .

dynamic function such as Equation (9), y_t is not sensitive to the volatility of i_t , because the function is a low pass filter that removes high frequencies and passes low frequencies of interest rates.

Figure 6 Actual Interest Rates and Trend of Interest Rates

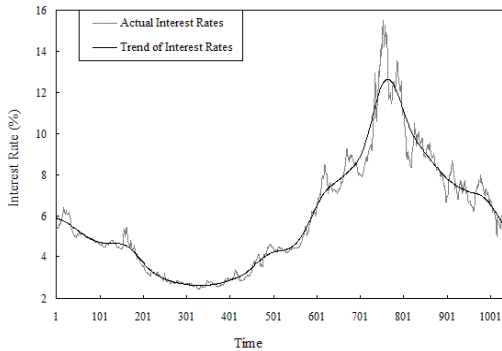


Figure 7 Present Values Computed by a Static Function and Two Types of Interest Rates

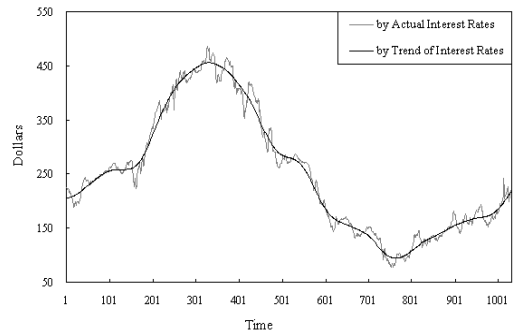


Figure 8 Present Values Computed by a Dynamic Function and Actual Interest Rates

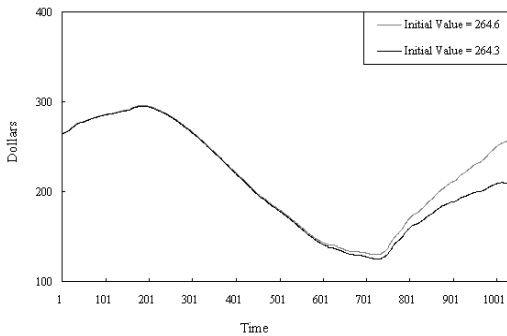
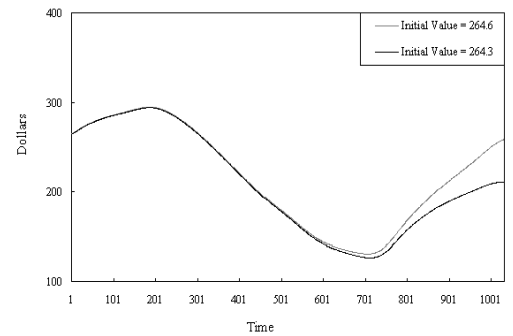


Figure 9 Present Values Computed by a Dynamic Function and Trend of Interest Rates



Inherently Static Stochastic Pricing Models

Since the interest rate models that employ stochastic differential equations are well developed and widely used, they provide the foundation of pricing and risk evaluation models in financial theory and related applications. The methodology of stochastic models is characterized by three features. First, the time series of interest rates is a stochastic process, and thus the time series of $PV(t)$ is a stochastic process. Second, stochastic calculus is employed to determine the distribution of $PV(t)$ based on the distributions of interest rates and default rates. Analytical solutions can be obtained under certain conditions. Third, some financial products are embedded with options, justifying the integration of option pricing techniques into traditional models.

Many well-known interest rate models can be nested within the following stochastic differential equation, and alternative models can be obtained by placing restrictions on the parameters. (Chan et al, 1992)

$$d i_u = (\alpha + \beta i_u) du + \sigma i_u^\gamma dw_u, \quad dw_u \sim N(0, du) \quad (12)$$

The discount function of a certain annuity with continuous fixed payments for n years can be stated as:

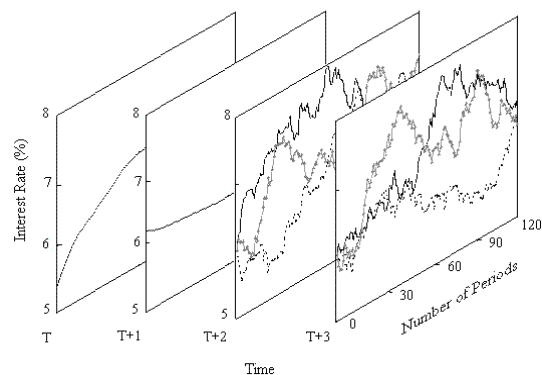
$$y_n = \int_0^n \exp\left(-\int_0^k i_u du\right) dk \quad (13)$$

The basic notion in stochastic modeling is that w_u is a determinant of the dynamics of i_u and y_n . Instantaneous changes in w_u drive instantaneous changes in both i_u and y_n , and the distribution of y_n can be deduced from those of w_u and i_u . In practice this methodology possesses the following two properties that make Equation (13) a static function.

- (1) In Equation (13), n represents payment duration rather than the time an event is observed. For example, y_{40} represents the present value of continuous payments for 40 years in the future regardless of the comparison date, while in the time series models y_{40} usually represents the value for the 40th year regardless of the payment duration. Equation (13) is static, because the concept of time is defined as the current time and the serial correlation between y_n for today and y_n for the next period is not a concern. Stating that the present values determined by a function such as Equation (13) represent a dynamic process is misleading. By contrast, in dynamic functions like Equation (7) and (9), t denotes the time of event observation; inputs of y_t and y_{t+1} are drawn from the same spot interest rate path, and the number of common inputs increases with the number of discounting, enhancing the correlation between y_t and y_{t+1} .
- (2) Methods of deriving interest rate paths comprise four categories: (a) a constant interest rate representing flat path level, (b) deterministic functions with parameters affecting the path shape, (c) stochastic models producing numerous and diverse paths, and (d) others, such as the lattice model and neural network method. These methods share a common characteristic of intertemporal independence of interest rate paths that makes y_n an independent random variable, which is an invisible assumption stating that the stationary process is equivalent to the cross section. Figure 10 illustrates this property. Figure 10 resembles a sequence of snapshots, each representing current market rates or a current forecast of future spot rates using deterministic functions or difference equations. In employing stochastic models and Monte Carlo simulations, simultaneous sampling of numerous independent w_u produces numerous independent interest rate paths in a snapshot, and periodic re-sampling of w_u produces the intertemporal independence of interest rate paths. Independence of y_n over time results from re-sampling of interest rate paths and recalculation of present values in response to changing market conditions.

Pragmatically the key difference between the static and dynamic discount functions lies in the sampling of discount rates. In the static function, all discount rates are renewed when the yield curve changes, making y_t sensitive to change in interest rates. In contrast, in the dynamic function the yield curve is not relevant, with only one or a few discount rates being replaced with new data and most of the discount rates being retained, generating autocorrelation for the time series of y_t .

Figure 10 Static Interest Rate Paths



CONCLUSION

This study investigates the empirical properties of discount functions and proposes that dynamic discount functions are low pass filters that remove high frequency noise associated with interest rates and yield a smooth curve of present values. Most stochastic models used in financial theory and practice are inherently static because of being based on equivalency between the stationary process and the cross section. Although the findings and theses of this study conflict with current financial theory, they are supported by filtering theory in mathematics. Recent decades have seen advances in applied mathematics and computer science transform financial markets at a speed that institutional adjustments are unable to keep up with, as well as facilitate the verification of the properties of discount functions with unexpected results. Over time continuing advances will hopefully facilitate further improvements in financial model construction.

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