A Deteriorating Two-Item Inventory Model with Continuously Decreasing Demand and Retroactive Holding Cost

Kuo-Hsien Wang, Department of Business Administration,
Takming University of Science and Technology ,Taipei , Taiwan
Che-Tsung Tung, Department of International Trade,
Takming University of Science and Technology, Taipei, Taiwan
Chen-Lin Chien, Department of International Trade,
Takming University of Science and Technology, Taipei, Taiwan
Feng-Chu Hung, Department of recreation and Sport Management,
Shu-Te University, Kaohsiung, Taiwan

ABSTRACT

Demand correlation among two interacting items is common in real-life markets. Sometimes demand of the first item would trigger additional demand toward the second item, but not vice versa. For items with decreasing demand in time, the longer is the selling period, the less the demand and the higher the inventory cost. This paper develops the related inventory model during which two deteriorating items are considered under the circumstances of retroactive increasing (decreasing) holding cost step function of storage time and two possible distinct selling periods as well. A sufficient condition to minimize total inventory cost per unit time is anticipated, via suitable control of the selling period. Ultimately, in response to the structure of holding cost, an efficient methodology for determining optimal selling periods is presented and demonstrated by numerical examples.

Keywords: *Inventory; Deteriorating; Holding cost; EOQ*

INTRODUCTION

Demand correlation among two interacting items is common in practice. In particular, a remarkable phenomenon frequently occurring is that demand of the first item would probably cause extra demand toward the second item, but not vice versa. Taking digital cameras and memory cards for example, customers who purchase digital cameras would be more likely to purchase memory cards at the same time, whereas those who are just buying memory cards would never purchase digital cameras as well. Similar comparisons include: PC vs. printer, printer vs. cartridge, cell phone vs. battery, and fashionable apparel vs. accessory etc. Specifically, a characteristic between them is that the second item usually has a longer selling period. Batteries are expected to be kept on the market for a period of time after the specific cell phone model stops selling.

Another widely discussed experience is that the demand of the first item would favor the demand of the second item, and vice versa. Bhattacharya (2005) investigated a two-item inventory model with a linear stock-dependent demand rate in a condition where mutual increase in demand is linearly correlated to the presence of the other item. Liu and Yuan (2000) constructed a Markovian model for a two-item inventory system with correlated demand and coordinated replenishments, whose demand was according to a Poisson process. Shah and Avittahur (2007) modeled a multi-item inventory problem with demand cannibalization and substitution, focusing on twin problems of optimal portfolio selection and optimal stocking.

Demand rate of product may vary throughout the selling period. Experimentally, fashionable apparel and consumer electronics are expected to sell better at the beginning of the selling period, while products such as holiday-linked products are expected to reach sales peak at the end of the selling period as the holiday approaches. A number of time-dependent demand rates have appeared in the literature, including Hariga (1993), Hariga and Benkherouf (1994), Urban and Baker (1997) and Khanra and Chaudhuri (2003): among them are linear, exponential, polynomial and quadratic forms of demand rate.

Most EOQ models have been postulated for various holding cost. Giri et al. (1996) developed a generalized model for deteriorating items: the holding cost of which is a continuous function of time. Weiss (1982) constructed a variation of EOQ model, whose holding cost is a convex function of time; that is in contrast to the classic model where holding cost is either a constant or a linear function of time. Goh (1994) considered two types of holding cost variation: nonlinear function of storage time and nonlinear function of storage level. Recently, Alfares (2007) first initiated holding cost as a step function of storage time into inventory system, in which two distinct step functions are introduced: retroactive and incremental increase.

The retroactive increasing (decreasing) step functions of holding cost are comprehensively applicable to practical environments. For items such as food products, the longer the products are stored, the more sophisticated the facilities and the more service needed, and hence the higher holding cost. On the contrary, the strategy of “the longer the storage time, the cheaper the holding cost” is usually being implemented to stimulate more demand from retailers.

Thus, unlike the Alfares (2007), we extend the holding cost step function structure to the two-item system. For each item, storage time can be partitioned off a series of time periods with distinctive unit holding costs that could be retroactive increasing (decreasing), which means that the unit holding cost at the last storage period will be applied to all previous storage periods. Also, the feature of the two items differs from those in Bhattacharya (2005) and Liu and Yuan (2000) by assuming (1) demands of items are continuously decreasing in time; (2) demand of the first item will trigger additional demand toward the second one, but not vice versa; and (3) two possible distinct selling periods are incorporated into our proposed model. In addition, the objective of this paper is to minimize the total inventory cost per unit time.

The remainder of this paper is organized as follows. Notation and assumptions are outlined in Section 2. We develop the inventory model and construct the objective function in Section 3 associated with theoretical analysis. In Section 4, a solution methodology is presented and demonstrated by numerical examples. Finally, conclusion and remarks on future research close the paper in Section 5.

Notation and Assumptions

The following notations will be used throughout this paper.

For item $i, i = 1, 2$, we clarify that:

$I_i(t) =$ inventory level at time *t*

$q_i =$ order quantity

$T_i =$ selling period, $T_1 \le T_2$

$\theta_i =$ constant unit deterioration rate per unit time

$h_i =$ unit holding cost per unit time

$c_i =$ unit purchasing cost

$f_i(t) =$ decreasing demand rate in time *t*

A deteriorating two-item inventory system starts operating at time $t = 0$ with the order quantities q_1 and q_2 for the first and the second item respectively, accompanied with the ordering cost *A*. During time span [0, T_1], the two items are sold concurrently with the assumed decreasing demand rates $f_1(t)$ and $f_2(t)$. For the second item, besides its own demand, there also exists the extra demand caused by the first item, whose amount is assumed to be in direct proportion to $f_1(t)$ with a fixed constant *r*, that is $r\, f_1(t)$. The first item will stop selling at T_1 with $I_1(T_1) = 0$,while the second item continues to sell on the following time span $[T_1, T_2]$. Eventually, the system ends at T_2 with $I_2(T_2)$=0. During the implementation of the system, shortage is not allowed, and for each item the unit holding cost at the last storage time period is applied to all its previous time period.

Model Formulation and Objective Function

According to the aforementioned assumptions, we have the following governed different equations for $I_1(t)$ and $I_2(t)$.

$$\frac{dI_1(t)}{dt} = -\theta_1 I_1(t) - f_1(t), \quad 0 < t < T_1 \tag{1}$$

$$\frac{dI_2(t)}{dt} = -\theta_2 I_2(t) - f_2(t) - rf_1(t), \quad 0 < t < T_1 \tag{2}$$

$$\frac{dI_2(t)}{dt} = -\theta_2 I_2(t) - f_2(t), \quad T_1 < t < T_2 \tag{3}$$

with the conditions $I_1(0) = q_1$, $I_2(0) = q_2$, $I_1(T_1) = 0$ and $I_2(T_2) = 0$.

Solutions for $I_1(t)$ and $I_2(t)$ are

$$I_1(t) = e^{-\theta_1 t}\int_t^{T_1} e^{\theta_1 u} f_1(u)du, \quad 0 \le t \le T_1 \tag{4}$$

$$I_2(t) = \begin{cases} e^{-\theta_2 t}\left(\int_t^{T_2} e^{\theta_2 u} f_2(u)du + \int_t^{T_1} re^{\theta_2 u} f_1(u)du\right), & 0 \le t \le T_1 \\ e^{-\theta_2 t}\int_t^{T_2} e^{\theta_2 u} f_2(u)du, & T_1 \le t \le T_2 \end{cases} \tag{5}$$

and

$$q_1 = \int_0^{T_1} e^{\theta_1 u} f_1(u)du \tag{6}$$

$$q_2 = \int_0^{T_2} e^{\theta_2 u} f_2(u)du + \int_0^{T_1} re^{\theta_2 u} f_1(u)du . \tag{7}$$

Objective Function

First, we construct the total inventory cost $TC(T_1, T_2)$ which is the sum of the ordering cost, holding cost and deteriorating cost.

$$\begin{aligned} TC(T_1,T_2) &= A + h_1\int_0^{T_1} I_1(t)dt + h_2\int_0^{T_2} I_2(t)dt + c_1\int_0^{T_1} \theta_1 I_1(t)dt + c_2\int_0^{T_2} \theta_2 I_2(t)dt \\ &= A + (h_1 + c_1\theta_1)\int_0^{T_1} I_1(t)dt + (h_2 + c_2\theta_2)\int_0^{T_2} I_2(t)dt \end{aligned} \tag{8}$$

Substituting (4),(5) into (8), and manipulating data, then

$$\begin{aligned} TC(T_1,T_2) = {} & A + (h_1 + c_1\theta_1)\int_0^{T_1} e^{-\theta_1 t}\int_t^{T_1} e^{\theta_1 u} f_1(u)dudt \\ & + (h_2 + c_2\theta_2)\left(\int_0^{T_1} e^{-\theta_2 t}\int_t^{T_1} re^{\theta_2 u} f_1(u)dudt + \int_0^{T_2} e^{-\theta_2 t}\int_t^{T_2} e^{\theta_2 u} f_2(u)dudt\right) \end{aligned} \tag{9}$$

Due to the unequal selling periods T_1, T_2 of the two correlated items, we define the total inventory cost per unit time, denoted by $\pi(T_1, T_2)$, as follows: dividing the total inventory cost by the real operation time unit of the system, that is

$$\pi(T_1,T_2) = \frac{TC(T_1,T_2)}{T_1 + T_2} \tag{10}$$

And this paper aims at finding the optimal selling periods T_1 and T_2 to minimize the $\pi(T_1, T_2)$.

Analysis

We will verify that $\pi(T_1, T_2)$ is a strictly pseudo convex function of T_1 and T_2. To this end, since the denominator is linear in T_1 and T_2, we just need to prove that the numerator $TC(T_1, T_2)$ is strictly convex function of

T_1 and T_2. Therefore, taking advantage of the formula:

$$\frac{d}{dT}\int_{\ell_1(T)}^{\ell_2(T)} g(T,t)dt = g(T,\ell_2(T)) \cdot \ell_2'(T) - g(T,\ell_1(T))\ell_1'(T) + \int_{\ell_1(T)}^{\ell_2(T)} \frac{\partial g(T,t)}{\partial T}dt$$, where $\ell'(\cdot)$ stands for the first-order derivative of $\ell(\cdot)$, the following first-order partial derivatives of $TC(T_1,T_2)$ with respect to T_1 and T_2 are obtained.

$$\frac{\partial TC(T_1,T_2)}{\partial T_1} = \frac{h_1 + c_1\theta_1}{\theta_1}\left(e^{\theta_1 T_1} - 1\right) f_1(T_1) + \frac{r(h_2 + c_2\theta_2)}{\theta_2}\left(e^{\theta_2 T_1} - 1\right) f_1(T_1) \quad (11)$$

$$\frac{\partial TC(T_1,T_2)}{\partial T_2} = \frac{h_2 + c_2\theta_2}{\theta_2}\left(e^{\theta_2 T_2} - 1\right) f_2(T_2) \quad (12)$$

For small values θ_1 and θ_2, the approximation $e^x - 1 = x$ is applied to (11) and (12), and it yields

$$\frac{\partial TC(T_1,T_2)}{\partial T_1} = \left(h_1 + c_1\theta_1 + rh_2 + rc_2\theta_2\right)T_1 f_1(T_1) \quad (13)$$

$$\frac{\partial TC(T_1,T_2)}{\partial T_2} = \left(h_2 + c_2\theta_2\right)T_2 f_2(T_2) \quad (14)$$

Under the circumstance, the second-order partial derivatives of $TC(T_1,T_2)$ are accordingly given by

$$\frac{\partial^2 TC(T_1,T_2)}{\partial T_1^2} = \left(h_1 + c_1\theta_1 + rh_2 + rc_2\theta_2\right)\left(f_1(T_1) + T_1 f_1'(T_1)\right) \quad (15)$$

$$\frac{\partial^2 TC(T_1,T_2)}{\partial T_2^2} = \left(h_2 + c_2\theta_2\right)\left(f_2(T_2) + T_2 f_2'(T_2)\right) \quad (16)$$

From the right-hand sides of (15) and (16), we find out that the values $\frac{\partial^2 TC(T_1,T_2)}{\partial T_i^2}$, $i = 1,2$, will be positive if the inequality $f_i(T_i) + T_i f_i'(T_i) > 0$ holds. And it is feasible because most existing time decreasing demands in the literature are satisfying the inequality. For instance, the demand $f(t) = \alpha t^{r-1}$, $\alpha > 0$, $0 < r < 1$, in Urban and Baker (1997) is unconditionally satisfying, also the demand $f(t) = \alpha e^{-\beta t}$, $\alpha > 0$, $\beta > 0$, in You and Chen (2007) is conditionally satisfying with constraint $T_i < \frac{1}{\beta}$ (β is usually small). Thus, through the suitable control of T_i, the positive $\frac{\partial^2 TC(T_1,T_2)}{\partial T_i^2}$ is anticipated, and thereby leads to the following outcome.

Theorem 1. For small θ_1 and θ_2, given decreasing $f_1(t)$ and $f_2(t)$, let $\mathring{T}_i = \sup\{T_i \mid f(T_i) + T_i f_i'(T_i) > 0\}$, $i = 1,2$, and $\mathring{T} = \min\{\mathring{T}_1, \mathring{T}_2\}$, then the $TC(T_1,T_2)$ is strictly convex on $J = \{(T_1,T_2) \mid 0 \le T_1 \le T_2 \le \mathring{T}\}$.

Proof. Based on the earlier analysis, $\frac{\partial^2 TC(T_1,T_2)}{\partial T_i^2} > 0$, $i = 1,2$ on *J*, also from (13), we have $\frac{\partial^2 TC(T_1,T_2)}{\partial T_1\,\partial T_2} = 0$.

Thus, the Hessian matrix $\begin{vmatrix} \frac{\partial^2 TC(T_1,T_2)}{\partial T_1^2} & \frac{\partial^2 TC(T_1,T_2)}{\partial T_1\,\partial T_2} \\ \frac{\partial^2 TC(T_1,T_2)}{\partial T_1\,\partial T_2} & \frac{\partial^2 TC(T_1,T_2)}{\partial T_2^2} \end{vmatrix} > 0$ on *J*, and this completes the proof.

Theorem 2. The $\pi(T_1,T_2)$ is strictly pseudo convex on *J*.

Proof. Combining the proven strict convexity of the numerator $TC(T_1,T_2)$ with the linear property of its denominator in T_1, T_2, this result is sequentially acquired. For more detail, the paper of Quyang et al. (2006) may be consulted.

Next, the selling periods to minimize $\pi(T_1,T_2)$ are concerned. If possible, the necessary optimality conditions $\frac{\partial \pi(T_1,T_2)}{\partial T_1}=0$ and $\frac{\partial \pi(T_1,T_2)}{\partial T_2}=0$ must be required simultaneously. Thus, we take the first-order partial derivatives of $\pi(T_1,T_2)$ with respective to T_1 and T_2, and set the result to be zero, then

$$\frac{\partial \pi(T_1,T_2)}{\partial T_1}=\frac{1}{(T_1+T_2)^2}\left(\frac{\partial TC(T_1,T_2)}{\partial T_1}(T_1+T_2)-TC(T_1,T_2)\right)$$
$$=\frac{1}{T_1+T_2}\left(\frac{\partial TC(T_1,T_2)}{\partial T_1}-\pi(T_1,T_2)\right)=0 \qquad (17)$$

$$\frac{\partial \pi(T_1,T_2)}{\partial T_2}=\frac{1}{(T_1+T_2)^2}\left(\frac{\partial TC(T_1,T_2)}{\partial T_2}(T_1+T_2)-TC(T_1,T_2)\right)$$
$$=\frac{1}{T_1+T_2}\left(\frac{\partial TC(T_1,T_2)}{\partial T_2}-\pi(T_1,T_2)\right)=0 \qquad (18)$$

Replacing $\pi(T_1,T_2)$ with $\frac{\partial TC(T_1,T_2)}{\partial T_2}$ in (17), and according to (11) and (12), we have

$$\frac{\partial \pi(T_1,T_2)}{\partial T_1}=\frac{1}{T_1+T_2}\left(\frac{\partial TC(T_1,T_2)}{\partial T_1}-\frac{\partial TC(T_1,T_2)}{\partial T_2}\right)$$
$$=\frac{1}{T_1+T_2}\left[\frac{h_1+c_1\theta_1}{\theta_1}\left(e^{\theta_1 T_1}-1\right)f_1(T_1)+\frac{r(h_2+c_2\theta_2)}{\theta_2}\left(e^{\theta_2 T_1}-1\right)f_1(T_1)\right.$$
$$\left.-\frac{h_2+c_2\theta_2}{\theta_2}\left(e^{\theta_2 T_2}-1\right)f_2(T_2)\right]=0 \qquad (19)$$

$$\frac{\partial \pi(T_1,T_2)}{\partial T_2}=\frac{1}{T_1+T_2}\left(\frac{\partial TC(T_1,T_2)}{\partial T_2}-\pi(T_1,T_2)\right)$$
$$=\frac{1}{T_1+T_2}\left[\frac{h_2+c_2\theta_2}{\theta_2}\left(e^{\theta_2 T_2}-1\right)f_2(T_2)-\pi(T_1,T_2)\right]=0 \qquad (20)$$

Theorem 3. For fixed T_2, there uniquely exists a $T_1 \in \left(0,T_2\right]$ so as to minimize $\pi(T_1,T_2)$.

Proof. Motivated by (19), let

$$F(T_1)=\frac{\partial TC(T_1,T_2)}{\partial T_1}-\frac{\partial TC(T_1,T_2)}{\partial T_2}$$
$$=\frac{h_1+c_1\theta_1}{\theta_1}\left(e^{\theta_1 T_1}-1\right)f_1(T_1)+\frac{r(h_2+c_2\theta_2)}{\theta_2}\left(e^{\theta_2 T_1}-1\right)f_1(T_1)$$
$$-\frac{h_2+c_2\theta_2}{\theta_2}\left(e^{\theta_2 T_2}-1\right)f_2(T_2) \qquad (21)$$

Then, we have $F'(T_1) = \dfrac{\partial^2 TC(T_1,T_2)}{\partial T_1^2} > 0$ on *J*, implying $F(T_1)$ is strictly increasing in T_1. Also, $F(0) = -\dfrac{h_2 + c_2\theta_2}{\theta_2}\left(e^{\theta_2 T_2} - 1\right) f_2(T_2) < 0$, hence if $F(T_2) > 0$ then T_1 is uniquely determined in $(0, T_2)$ such that (19) holds. Contrarily, if $F(T_2) \le 0$, according to (19), then $\dfrac{\partial \pi(T_1,T_2)}{\partial T_1} \le 0$. That way we take $T_1 = T_2$ to a purpose of minimizing $\pi(T_1,T_2)$, and this completes the proof.

Theorem 4. There always exists a unique $T_2 < \tilde{T}_2$ such that (20) holds.

Proof. First let $T_2 \to 0^+$, then $T_1 \to 0^+$, too. Also from (9) $TC(T_1,T_2)$ will approach to *A* as $T_2 \to 0^+$, so $\lim_{T_2 \to 0^+} \pi(T_1,T_2) = \lim_{T_2 \to 0^+} \dfrac{TC(T_1,T_2)}{T_1 + T_2} = \infty$.

Next, because the $\pi(T_1,T_2)$ is strictly convex for $T_2 < \tilde{T}_2$ and strictly concave for $T_2 > \tilde{T}_2$, an inflection point occurs at $\tilde{T}_2$. Therefore, three possible behaviors of $\pi(T_1,T_2)$ with respect to T_2 are concluded in Fig.1.

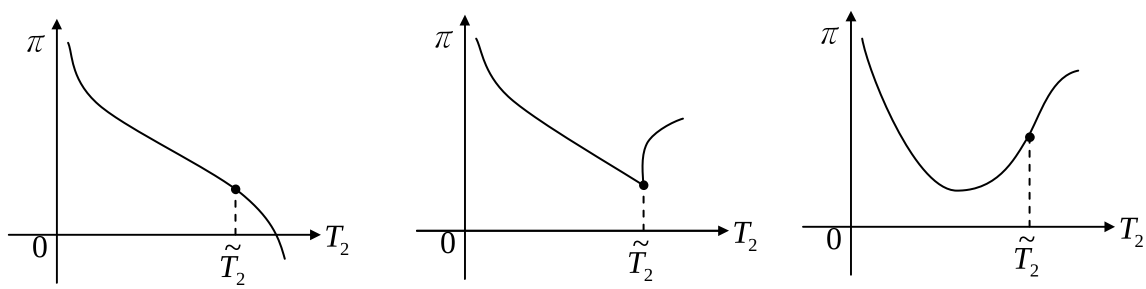

Figure 1: Three possible behaviors of π with respect to T_2

Obviously, the first behavior contradicts the positive value of $\pi(T_1,T_2)$, and the second violates the smooth function of $\pi(T_1,T_2)$ at $\tilde{T}_2$. Only the last fits the properties of $\pi(T_1,T_2)$, which consequently results in a horizontal tangent line in $(0, \tilde{T}_2)$ and this completes the proof.

Solution Methodology and Examples

According to the previous section, we have learned the optimal solution T_1^* and T_2^* can be obtained either by solving (19) and (20) simultaneously with the fact that $0 < T_1^* < T_2^* < \tilde{T}_2$, or by solving (20) alone in response to $0 < T_1^* = T_2^* < \tilde{T}_2$. In dealing with the step function structure of holding cost, we still need to find T_1^*, T_2^* that match with the given corresponding h_1, h_2. With the aid of graphics, the following two examples will help illustrate our methodology.

Example 1. (Retroactive increasing) Given the following parameters: $A = 500$, $c_1 = 10$, $c_2 = 5$, $\theta_1 = 0.01$, $\theta_2 = 0.01$, $r = 0.2$, $f_1(t) = 300e^{-0.003t}$, $f_2(t) = 450e^{-0.002t}$, and the unit holding costs are listed below. (Note that $\tilde{T}_1 = \dfrac{1000}{3}$, $\tilde{T}_2 = 500$, and $\tilde{T} = \dfrac{1000}{3}$)

h_1	time period	h_2
8	$(0, 0.2]$	4
9	$(0.2, 0.4]$	5
10	$(0.4, \infty)$	6

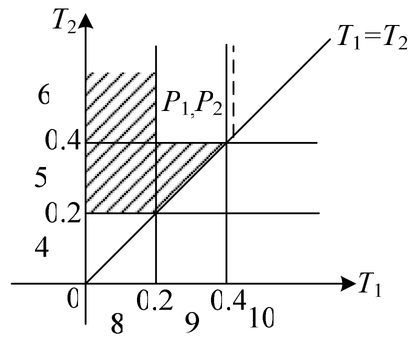

Figure 2: Increasing holding cost

Before proceeding, for convenience, we designate $H(h_1, h_2)$ as a pair of values h_1, h_2 with the corresponding domain given. Taking $H(8,5)$ for example, $h_1 = 8$, $h_2 = 5$, and $(0, 0.2] \times (0.2, 0, 4]$ is its domain. Also, we define $P(T_1, T_2 | H(h_1, h_2))$ as a solution point to (19), (20) under the value $H(h_1, h_2)$ and as a realizable solution point if $P(T_1, T_2 | H(h_1, h_2)) \in H(h_1, h_2)$.

Step 1. Find the realizable solution point

Starting with *H*(8,4) to solve (19),(20), the solution point obtained is $P_1(0.3893, 0.5705 | H(8,4)) \in H(9,6)$ which is not realizable. Thus applying the *H*(9,6) to solve (19),(20) again, the solution point obtained is $P_2(0.3889, 0.4416 | H(9,6)) \in H(9,6)$ which is realizable (see Fig.2), so P_2 is the realizable solution point as well as a possible optimal solution to this problem.

Step 2. Find other possible optimal solutions

Other than $P_2(0.3889, 0.4416 | H(9,6))$, other possible optimal solutions are expected to arise at locations adjacent to *H*(9,6), including *H*(8,6), *H*(8,5) and *H*(9,5) (shady regions in Fig.2), but excluding *H*(10,6) because of its left-hand solution point and left open boundary, also excluding *H*(10,5) because of the requirement $T_1 \le T_2$. Therefore, the following movements are taken to find other possible optimal solutions.

Applying *H*(8,6) to (19) and (20) to obtain a solution point $P(0.4202, 0.4309 | H(8,6)) \in H(10,6)$ which is not realizable, thus the boundary point $(0.2, 0.4309 | H(8,6))$ nearest to *P* is taken as a possible optimal solution.

Applying *H*(8,5) to (19) and (20) to obtain a solution point $Q(0.4074, 0.4896 | H(8,5)) \in H(10,6)$ which is not realizable, thus the boundary point $(0.2, 0.4 | H(8,5))$ nearest to *Q* is taken as a possible optimal solution.

Applying *H*(9,5) to (19) and (20) to obtain a solution point $R(0.3757, 0.5010 | H(9,5)) \in H(9,6)$ which is not realizable, thus the boundary point $(0.3757, 0.4 | H(9,5))$ nearest to *R* is chosen as a possible optimal solution.

Step 3. Calculate the corresponding value π by (10)

$$\pi\left(0.3889, 0.4416 \middle| H(9,6)\right) = 1203.75$$

$$\pi\left(0.2, 0.4309 \middle| H(8,6)\right) = 1282.05$$

$$\pi\left(0.2, 0.4 \middle| H(8,5)\right) = 1227.70$$

$$\pi\left(0.3757, 0.4 \middle| H(9,5)\right) = 1155.23$$

As a result, the optimal values to the problem are $T_1^* = 0.3757$, $T_2^* = 0.4$, $\pi^* = 1155.23$, $q_1^* = 112.847$ and $q_2^* = 202.858$.

Example 2. (Retroactive decreasing) The same parameters as Example 1, except for the unit holding costs as given below:

h_1	time period	h_2
11	$(0, 0.2]$	7
10	$(0.2, 0.4]$	6
9	$(0.4, \infty)$	5

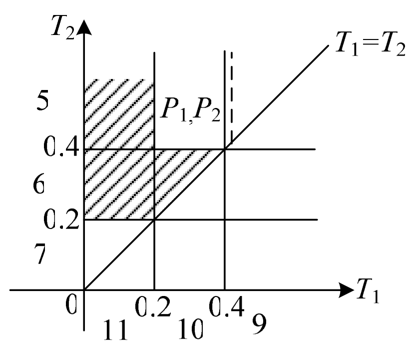

Figure 3: Decreasing holding cost

Step 1. Find the realizable solution point

Applying *H*(11,7) to (19) and (20), to obtain $P_1\left(0.3492, 0.4129 \middle| H(11,7)\right) \in H(10,5)$, again applying *H*(10,5) to (19) and (20) to obtain $P_2\left(0.3487, 0.5109 \middle| H(10,5)\right) \in H(10,5)$, thus P_2 is the realizable solution point and a possible optimal solution as well.

Step 2. Find other possible optimal solutions

Applying *H*(11,5) to (19) and (20) to obtain $\left(0.3254, 0.5197 \middle| H(11,5)\right) \in H(10,5)$, thus the boundary point $\left(0.2, 0.5197 \middle| H(11,5)\right)$ is chosen.

Applying *H*(11,6) to (19) and (20) to obtain $\left(0.3389, 0.4594 \middle| H(11,6)\right) \in H(10,5)$, thus the boundary point $\left(0.2, 0.4 \middle| H(11,6)\right)$ is chosen.

Applying *H*(10,6) to (19) and (20) to obtain $\left(0.3621, 0.4510 \middle| H(10,6)\right) \in H(10,5)$, thus the boundary point $\left(0.3621, 0.4 \middle| H(10,6)\right)$ is chosen.

Step 3. Calculate the corresponding value π

$$\pi\left(0.3487, 0.5109 \middle| H(10,5)\right) = 1162.88$$

$$\pi\left(0.2, 0.5197 \middle| H(11,5)\right) = 1222.57$$

$$\pi\left(0.2, 0.4 \middle| H(11,6)\right) = 1247.71$$

$$\pi\left(0.3621, 0.4 \middle| H(10,6)\right) = 1234.12$$

As a result, the optimal values to the problem are $T_1^* = 0.3487$, $T_2^* = 0.5109$, $\pi^* = 1162.88$, $q_1^* = 104.725$ and $q_2^* = 251.336$.

CONCLUSION

During the solution-finding process, the realizable solution point which also is a possible optimal solution needs to be found first, so as to prompt other possible optimal solutions to be obtained accordingly. However, no theory supports the must-existence of the realizable solution point, though the two numerical examples both show it indeed exists. One possible situation that cannot be ruled out is that: at Step 1, the solution points would never converge to the realizable solution point, but instead they may oscillate between two regions in the end. Under the circumstance, the possible optimal solutions located adjacent to the two regions are necessary to be considered altogether. The proposed model can be applied to any non-decreasing time demand rates since the positive values of (15) and (16) automatically satisfy the equations. Also, a multi-item system could fit well into our solution methodology. For future research, the proposed model can be generalized to a more complicated demand rate such as multivariate function of time, selling price and instantaneous inventory level.

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